

All joint von Neumann measurements on a quantum state admit a quasi-classical probability model

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Abstract

We prove that the Hilbert space description of *all* joint von Neumann measurements on a quantum state can be reproduced in terms of a *single* measure space $(\Omega, \mathcal{F}_\Omega, \mu)$ with a normalized real-valued measure μ , that is, in terms of a new general probability model, *the quasi-classical probability model*, developed in [Loubenets: *J. Math. Phys.* 53 (2012), 022201; *J. Phys. A: Math. Theor.* 45 (2012), 185306]. In a quasi-classical probability model for all von Neumann measurements, a random variable models the corresponding quantum observable in all joint measurements and depends only on this quantum observable. This mathematical result sheds a new light on some important issues of quantum randomness discussed in the literature since the seminal article (1935) of Einstein, Podolsky and Rosen.

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1 Introduction

The relation between the quantum probability model and the classical probability model has been a point of intensive discussions ever since the seminal publications of von Neumann [1],

Kolmogorov [2] and Einstein, Podolsky and Rosen (EPR) [3].

Though, in the quantum physics literature, one can still find the misleading claims on a peculiarity of "quantum probabilities" and "quantum events", the probabilistic description of every quantum measurement, generalized [4] or projective, satisfies the Kolmogorov axioms [2] for the theory of probability. For example, the von Neumann measurement of a quantum observable X in a state ρ on a complex separable Hilbert space \mathcal{H} is described by the probability space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \text{tr}[\rho P_X(\cdot)])$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ -algebra on \mathbb{R} and P_X is the spectral projection-valued measure on $\mathcal{B}_{\mathbb{R}}$ uniquely corresponding to an observable X on \mathcal{H} due to the spectral theorem [1, 5].

However, the Hilbert space description of *all* joint von Neumann measurements on an arbitrary quantum state cannot be reproduced in terms of a single probability space. The same concerns the probabilistic description of an arbitrary quantum multipartite correlation scenario with finite numbers of settings at each site and, more generally, an arbitrary nonsignaling multipartite correlation scenario, see introductions in [6, 7, 8] and references therein.

Note that, in the quantum theory literature, the interpretation of quantum measurements via the classical probability model is generally referred to as a hidden variable (HV) model, and a local hidden variable (LHV) model constitutes a version of an HV model, where a random variable modelling a marginal measurement depends only on a setting of this measurement.

Analyzing the probabilistic description of a *general* multipartite correlation scenario with a finite number of settings at each site, we have introduced [7] the notion of a *local quasi hidden variable (LqHV) model*, where locality and the measure-theoretic structure inherent to a local hidden variable (LHV) model are preserved but positivity of a simulation measure is dropped. We have proved [7] that every quantum N -partite correlation scenario admits an LqHV model.

Developing the LqHV approach further, we showed [8] that a general correlation scenario admits an LqHV model (i) if and only if it is nonsignaling [6] and (ii) if and only if it admits a deterministic LqHV model. In the latter particular type of an LqHV model, all joint probability distributions of a correlation scenario are reproduced in terms of a single measure space $(\Omega, \mathcal{F}_{\Omega}, \mu)$ with a normalized bounded measure μ and a set of random variables each depending only on a setting of the corresponding modelled marginal measurement.

As we have argued in [8], these new results point to the existence of a new general probability model, the *quasi-classical probability model*, that has the measure-theoretic structure $(\Omega, \mathcal{F}_{\Omega}, \nu)$ resembling the structure of the classical probability model but reduces to the latter iff a normalized real-valued measure ν is positive.

In the present paper, we prove that the Hilbert space description of *all joint von Neumann measurements* on a quantum system can be reproduced in terms of a *single* space $(\Omega, \mathcal{F}_{\Omega})$ via a set of normalized real-valued measures, each uniquely corresponding to a modelled quantum state, and a set of random variables, each modelling the corresponding quantum observable in all joint von Neumann measurements and depending only on this quantum observable.

This result, in particular, means that the probabilistic description of all joint von Neumann measurements on a quantum state admits the quasi-classical probability model introduced in [8] and, in this model, a random variable modelling a quantum observable in all joint von Neumann measurements is determined only by this quantum observable.

The paper is organized as follows.

In section 2, we recall the von Neumann formalism for the description of ideal quantum measurements and the notion of the spectral projection-valued measure of a quantum

observable.

In section 3, we introduce for a finite number of quantum observables the symmetrized product of their spectral measures and discuss properties of this product operator-valued measure.

In section 4, we generalize some items of the Kolmogorov extension theorem [2] to the case of consistent operator-valued measures and specify this generalization for the consistent product measures introduced in section 3.

In section 5, we prove that the Hilbert space description of all joint von Neumann measurements on a quantum system can be reproduced in terms of random variables and normalized real-valued measures defined on a single measurable space.

In section 6, we summarize the main mathematical results of the present paper and discuss their conceptual implications.

2 Von Neumann measurements

In the frame of the von Neumann approach [1], states and observables of a quantum system are described, correspondingly, by density operators ρ and self-adjoint linear operators X , bounded or unbounded, on a complex separable Hilbert space \mathcal{H} .

Denote by $\mathfrak{X}_{\mathcal{H}}$ the set of all self-adjoint linear operators, bounded and unbounded, on \mathcal{H} . Let $\mathcal{L}_{\mathcal{H}}$ be the vector space of all bounded linear operators on \mathcal{H} and $\mathcal{L}_{\mathcal{H}}^{(s)}$ – the vector space of all self-adjoint bounded linear operators on \mathcal{H} . Equipped with the operator norm, these vector spaces are Banach.

The probability that, under an ideal (errorless) measurement of a quantum observable $X \in \mathfrak{X}_{\mathcal{H}}$ in a state ρ , an observed value belongs to a Borel subset B of \mathbb{R} is given [1, 4, 5] by the expression

$$\mathrm{tr}[\rho P_X(B)], \quad B \in \mathcal{B}_{\mathbb{R}}, \quad (1)$$

where $\mathcal{B}_{\mathbb{R}}$ is the σ -algebra [9] of Borel subsets of \mathbb{R} and P_X is the spectral projection-valued measure of a quantum observable X – that is, the measure P_X on $\mathcal{B}_{\mathbb{R}}$ uniquely corresponding to $X \in \mathfrak{X}_{\mathcal{H}}$ due to the spectral theorem [1, 5, 10]

$$X = \int_{\mathbb{R}} \lambda P_X(d\lambda) \quad (2)$$

and with values $P_X(B)$, $\forall B \in \mathcal{B}_{\mathbb{R}}$, $P_X(\mathbb{R}) = \mathbb{I}_{\mathcal{H}}$, that are projections on \mathcal{H} satisfying the relations

$$\begin{aligned} P_X(B_1)P_X(B_2) &= P_X(B_2)P_X(B_1) = P_X(B_1 \cap B_2), \quad B_1, B_2 \in \mathcal{B}_{\mathbb{R}}, \\ P_X(B) &= 0, \quad \text{iff } B \in \mathcal{B}_{\mathbb{R}} \cap (\mathbb{R}/\mathrm{sp}X), \end{aligned} \quad (3)$$

where the spectrum $\mathrm{sp}X$ of an observable $X \in \mathfrak{X}_{\mathcal{H}}$ constitutes a closed Borel subset of \mathbb{R} .

An ideal measurement (1) of a quantum observable X in a state ρ is generally referred to as the von Neumann measurement.

The measure P_X is σ -additive in the strong operator topology [4, 5, 10] in $\mathcal{L}_{\mathcal{H}}^{(s)}$, that is:

$$\lim_{n \rightarrow \infty} \left\| P_X(\cup_{i=1}^{\infty} B_i)\psi - \sum_{i=1}^n P_X(B_i)\psi \right\|_{\mathcal{H}} = 0 \quad (4)$$

for all $\psi \in \mathcal{H}$ and all countable collections $\{B_i\}$ of mutually disjoint sets in $\mathcal{B}_{\mathbb{R}}$.

Remark 1 In this article, we follow the terminology of Ref. [11]. Namely, let \mathfrak{B} be a Banach space and \mathcal{F}_Λ be an algebra of subsets of a set Λ . We refer to an additive set function $\mathfrak{m} : \mathcal{F}_\Lambda \rightarrow \mathfrak{B}$ as a \mathfrak{B} -valued (finitely additive) measure on \mathcal{F}_Λ . If a measure \mathfrak{m} on \mathcal{F}_Λ is σ -additive in some topology on \mathfrak{B} , then we specify this in addition.

From (3) it follows that, for each $X \in \mathfrak{X}_\mathcal{H}$, the measure $P_X(B) \neq 0$ if and only if a set $B \neq \emptyset$ belongs to the trace σ -algebra

$$\mathcal{B}_{\text{sp}X} := \mathcal{B}_\mathbb{R} \cap \text{sp}X. \quad (5)$$

Therefore, we further consider the spectral projection-valued measure P_X only on $\mathcal{B}_{\text{sp}X}$.

The joint von Neumann measurement of several quantum observables $X_1, \dots, X_n \in \mathfrak{X}_\mathcal{H}$ is possible [1, 4, 5] iff all values of their spectral measures mutually commute, that is,

$$[P_{X_{i_1}}(B_{i_1}), P_{X_{i_2}}(B_{i_2})] = 0, \quad B_i \in \mathcal{B}_{\text{sp}X_i}, \quad i = 1, \dots, n. \quad (6)$$

For bounded quantum observables $X_1, \dots, X_n \in \mathfrak{X}_\mathcal{H}$, condition (6) is equivalent to mutual commutativity $[X_{i_1}, X_{i_2}] = 0, i = 1, \dots, n$, of these observables. Therefore, for short, we further refer to arbitrary quantum observables $X_1, \dots, X_n \in \mathfrak{X}_\mathcal{H}$, for which the spectral measures satisfy condition (6), as *mutually commuting* in the sense of condition (6).

The joint von Neumann measurement of *mutually commuting* quantum observables $X_1, \dots, X_n \in \mathfrak{X}_\mathcal{H}$ is described [1, 4, 5] by the normalized projection-valued measure

$$\int_{(\lambda_1, \dots, \lambda_n) \in B} P_{X_1}(d\lambda_1) \cdot \dots \cdot P_{X_n}(d\lambda_n), \quad B \in \mathcal{B}_{\text{sp}X_1 \times \dots \times \text{sp}X_n}, \quad (7)$$

on the trace Borel σ -algebra

$$\mathcal{B}_{\text{sp}X_1 \times \dots \times \text{sp}X_n} := \mathcal{B}_\mathbb{R}^n \cap (\text{sp}X_1 \times \dots \times \text{sp}X_n). \quad (8)$$

This measure is [10] σ -additive in the strong operator topology on $\mathcal{L}_\mathcal{H}^{(s)}$.

The expression

$$\text{tr}[\rho\{P_{X_1}(B_1) \cdot \dots \cdot P_{X_n}(B_n)\}] \quad (9)$$

gives the probability that, under the joint von Neumann measurement of mutually commuting quantum observables $X_1, \dots, X_n \in \mathfrak{X}_\mathcal{H}$ in a state ρ , these observables take values in sets $B_1 \in \mathcal{B}_{\text{sp}X_1}, \dots, B_n \in \mathcal{B}_{\text{sp}X_n}$, respectively.

3 Symmetrized products of spectral measures

For an n -tuple (X_1, \dots, X_n) of arbitrary mutually non-equal observables $X_1, \dots, X_n \in \mathfrak{X}_\mathcal{H}$, consider on the set $\text{sp}X_1 \times \dots \times \text{sp}X_n \subseteq \mathbb{R}^n$ the algebra $\mathcal{F}_{\text{sp}X_1 \times \dots \times \text{sp}X_n}$, the *product algebra*, generated by all rectangles $B_1 \times \dots \times B_n \subseteq \text{sp}X_1 \times \dots \times \text{sp}X_n$ with measurable sides $B_i \in \mathcal{B}_{\text{sp}X_i}$.

Let

$$\mathcal{P}_{(X_1, \dots, X_n)} : \mathcal{F}_{\text{sp}X_1 \times \dots \times \text{sp}X_n} \rightarrow \mathcal{L}_\mathcal{H}^{(s)} \quad (10)$$

be the normalized finitely additive $\mathcal{L}_\mathcal{H}^{(s)}$ -valued measure defined *uniquely* on $\mathcal{F}_{\text{sp}X_1 \times \dots \times \text{sp}X_n}$ via the relation

$$\mathcal{P}_{(X_1, \dots, X_n)}(B_1 \times \dots \times B_n) = \frac{1}{n!} \{P_{X_1}(B_1) \cdot \dots \cdot P_{X_n}(B_n)\}_{\text{sym}} \quad (11)$$

on all rectangles $B_1 \times \cdots \times B_n$ with sides $B_i \in \mathcal{B}_{\text{sp}X_i}$. Here, the notation $\{Z_1 \cdots Z_n\}_{\text{sym}}$ means the sum constituting the symmetrization of the operator product $Z_1 \cdots Z_n$, $Z_i \in \mathcal{L}_{\mathcal{H}}^{(s)}$, with respect to all permutations of its factors.

If each observable X_i in a collection $\{X_1, \dots, X_n\} \subset \mathfrak{X}_{\mathcal{H}}$ is bounded (i.e. $X_i \in \mathcal{L}_{\mathcal{H}}^{(s)}$) and has only a discrete spectrum $\text{sp}X_i = \{\lambda_{X_i}^{(k)} \in \mathbb{R}, k = 1, \dots, K_{X_i} < \infty\}$, where $\lambda_{X_i}^{(k)}$ are eigenvalues of X_i , then the product algebra $\mathcal{F}_{\text{sp}X_1 \times \cdots \times \text{sp}X_n}$ is finite and coincides with the Borel algebra $\mathcal{B}_{\text{sp}X_1 \times \cdots \times \text{sp}X_n}$, and the finitely additive measure $\mathcal{P}_{(X_1, \dots, X_n)}$ has the form¹

$$\mathcal{P}_{(X_1, \dots, X_n)}(F) := \frac{1}{n!} \sum_{(\lambda_{X_1}, \dots, \lambda_{X_n}) \in F} \{P_{X_1}(\{\lambda_{X_1}\}) \cdots P_{X_n}(\{\lambda_{X_n}\})\}_{\text{sym}} \quad (12)$$

for all $F \in \mathcal{F}_{\text{sp}X_1 \times \cdots \times \text{sp}X_n}$.

For quantum observables $X_1, \dots, X_n \in \mathfrak{X}_{\mathcal{H}}$, which mutually commute in the sense of relation (6), the measure $\mathcal{P}_{(X_1, \dots, X_n)}$ is projection-valued and $\|\mathcal{P}_{(X_1, \dots, X_n)}(F)\| = 1$ for each $\emptyset \neq F \in \mathcal{F}_{\text{sp}X_1 \times \cdots \times \text{sp}X_n}$.

Consider the family

$$\{\mathcal{P}_{(X_1, \dots, X_n)} : \mathcal{F}_{\text{sp}X_1 \times \cdots \times \text{sp}X_n} \rightarrow \mathcal{L}_{\mathcal{H}}^{(s)} \mid \{X_1, \dots, X_n\} \subset \mathfrak{X}_{\mathcal{H}}, n \in \mathbb{N}\} \quad (13)$$

of all normalized finitely additive $\mathcal{L}_{\mathcal{H}}^{(s)}$ -valued measures (10). These measures satisfy the following relations proved in appendix A.

Lemma 1 *For an arbitrary finite collection $\{X_1, \dots, X_n\} \subset \mathfrak{X}_{\mathcal{H}}$ of quantum observables on \mathcal{H} ,*

$$\begin{aligned} \mathcal{P}_{(X_1, \dots, X_n)}(B_1 \times \cdots \times B_n) &= \mathcal{P}_{(X_{i_1}, \dots, X_{i_n})}(B_{i_1} \times \cdots \times B_{i_n}), \\ B_i &\in \mathcal{B}_{\text{sp}X_i}, \quad i = 1, \dots, n, \end{aligned} \quad (14)$$

for all permutations $\binom{1, \dots, n}{i_1, \dots, i_n}$ and

$$\begin{aligned} &\mathcal{P}_{(X_1, \dots, X_n)}(\{(x_1, \dots, x_n) \in \text{sp}X_1 \times \cdots \times \text{sp}X_n \mid (x_{i_1}, \dots, x_{i_k}) \in F\}) \\ &= \mathcal{P}_{(X_{i_1}, \dots, X_{i_k})}(F), \quad F \in \mathcal{F}_{\text{sp}X_{i_1} \times \cdots \times \text{sp}X_{i_k}}, \end{aligned} \quad (15)$$

for each subset $\{X_{i_1}, \dots, X_{i_k}\} \subseteq \{X_1, \dots, X_n\}$.

Relations (14), (15) on operator-valued measures $\mathcal{P}_{(X_1, \dots, X_n)}$ are quite similar by their form to the Kolmogorov consistency conditions [2, 12] for a family

$$\{\mu_{(t_1, \dots, t_n)} : \mathcal{B}_{\mathbb{R}^n} \rightarrow [0, 1] \mid \{t_1, \dots, t_n\} \subset T, n \in \mathbb{N}\} \quad (16)$$

of probability measures $\mu_{(t_1, \dots, t_n)}$, each specified by a tuple (t_1, \dots, t_n) of mutually non-equal elements in an index set T .

In view of this similarity, for our further consideration in section 5, we proceed to generalize to the case of consistent operator-valued measures some items of the Kolmogorov theorem [2] on the extension to $(\mathbb{R}^T, \mathcal{F}_{\mathbb{R}^T})$ of consistent probability measures (16).

Remark 2 *Notations \mathbb{R}^T and $\mathcal{F}_{\mathbb{R}^T}$ mean [2, 12], correspondingly, the set of all real-valued functions $x : T \rightarrow \mathbb{R}$ and the algebra generated on \mathbb{R}^T by all cylindrical subsets of the form $\{x \in \mathbb{R}^T \mid (x(t_1), \dots, x(t_n)) \in B\}$, where $B \in \mathcal{B}_{\mathbb{R}^n}$, $\{t_1, \dots, t_n\} \subset T$, $n \in \mathbb{N}$.*

¹Here, the generally accepted notation $\sum_{\lambda \in B} Z(\lambda)$ means the sum $\sum_{\lambda} \chi_B(\lambda) Z(\lambda)$, where $\chi_B(\cdot)$ is the indicator function of a set B , that is, $\chi_B(\lambda) = 1$ for $\lambda \in B$ and $\chi_B(\lambda) = 0$ for $\lambda \notin B$.

4 The extension theorem

For an uncountable index set T , consider a family $\{(\Lambda_t, \mathcal{F}_{\Lambda_t}), t \in T\}$, where each Λ_t is a non-empty set and \mathcal{F}_{Λ_t} is an algebra of subsets of Λ_t . Let $\mathcal{F}_{\Lambda_{t_1} \times \dots \times \Lambda_{t_n}}$ be the algebra on $\Lambda_{t_1} \times \dots \times \Lambda_{t_n}$, the *product algebra*, generated by all rectangles $F_1 \times \dots \times F_n \subseteq \Lambda_{t_1} \times \dots \times \Lambda_{t_n}$ with sides $F_k \in \mathcal{F}_{\Lambda_{t_k}}$.

Denote by $\Lambda := \prod_{t \in T} \Lambda_t$ the Cartesian product [9] of all sets Λ_t , $t \in T$. That is, Λ is the collection of all functions $\lambda : T \rightarrow \cup_{t \in T} \Lambda_t$ with values $\lambda_t := \lambda(t) \in \Lambda_t$.

The set of all cylindrical subsets of Λ of the form

$$\begin{aligned} \mathcal{J}_{(t_1, \dots, t_n)}(F) &: = \{\lambda \in \Lambda \mid (\lambda_{t_1}, \dots, \lambda_{t_n}) \in F\}, \\ F &\in \mathcal{F}_{\Lambda_{t_1} \times \dots \times \Lambda_{t_n}}, \quad \{t_1, \dots, t_n\} \subset T, \quad n \in \mathbb{N}, \end{aligned} \quad (17)$$

constitutes [9] an algebra on Λ that we further denote by \mathcal{A}_Λ .

Since $\mathcal{J}_{(t_1, \dots, t_n)}(F) \equiv \pi_{(t_1, \dots, t_n)}^{-1}(F)$, where the function $\pi_{(t_1, \dots, t_n)} : \Lambda \rightarrow \Lambda_{t_1} \times \dots \times \Lambda_{t_n}$ is the canonical projection on Λ defined by the relations

$$\begin{aligned} \pi_{(t_1, \dots, t_n)}(\lambda) &: = (\pi_{t_1}(\lambda), \dots, \pi_{t_n}(\lambda)) \in \Lambda_{t_1} \times \dots \times \Lambda_{t_n}, \\ \pi_t(\lambda) &: = \lambda_t \in \Lambda_t, \end{aligned} \quad (18)$$

we have

$$\mathcal{A}_\Lambda = \{\pi_{(t_1, \dots, t_n)}^{-1}(F) \subseteq \Lambda \mid F \in \mathcal{F}_{\Lambda_{t_1} \times \dots \times \Lambda_{t_n}}, \quad \{t_1, \dots, t_n\} \subset T, \quad n \in \mathbb{N}\}. \quad (19)$$

Introduce a family

$$\{\mathfrak{M}_{(t_1, \dots, t_n)} : \mathcal{F}_{\Lambda_{t_1} \times \dots \times \Lambda_{t_n}} \rightarrow \mathcal{L}_{\mathcal{H}} \mid \mathfrak{M}_{(t_1, \dots, t_n)}(\Lambda_{t_1} \times \dots \times \Lambda_{t_n}) = \mathbb{I}_{\mathcal{H}}, \quad \{t_1, \dots, t_n\} \subset T, \quad n \in \mathbb{N}\} \quad (20)$$

of normalized finitely additive measures $\mathfrak{M}_{(t_1, \dots, t_n)}$, each specified by mutually non-equal indices $t_1, \dots, t_n \in T$ and having values that are bounded linear operators on \mathcal{H} .

Let, for each finite index collection $\{t_1, \dots, t_n\} \subset T$, these measures satisfy the consistency condition

$$\begin{aligned} \mathfrak{M}_{(t_1, \dots, t_n)}(F_1 \times \dots \times F_n) &= \mathfrak{M}_{(t_{i_1}, \dots, t_{i_n})}(F_{i_1} \times \dots \times F_{i_n}), \\ F_i &\in \mathcal{F}_{\Lambda_{t_i}}, \quad i = 1, \dots, n, \end{aligned} \quad (21)$$

for all permutations $(i_1, \dots, i_n)^{(1, \dots, n)}$ and the consistency condition

$$\begin{aligned} &\mathfrak{M}_{(t_1, \dots, t_n)}(\{(\lambda_1, \dots, \lambda_n) \in \Lambda_{t_1} \times \dots \times \Lambda_{t_n} \mid (\lambda_{i_1}, \dots, \lambda_{i_k}) \in F\}) \\ &= \mathfrak{M}_{(t_{i_1}, \dots, t_{i_k})}(F), \quad F \in \mathcal{F}_{\Lambda_{t_{i_1}} \times \dots \times \Lambda_{t_{i_k}}}, \end{aligned} \quad (22)$$

for each $\{t_{i_1}, \dots, t_{i_k}\} \subseteq \{t_1, \dots, t_n\}$.

The following statement is proved in appendix B and constitutes a generalization to the case of consistent operator-valued measures of some items of the Kolmogorov consistency theorem [2, 12] for probability measures (16).

Lemma 2 For a family (20) of normalized finitely additive $\mathcal{L}_{\mathcal{H}}$ -valued measures $\mathfrak{M}_{(t_{i_1}, \dots, t_{i_n})}$ satisfying the consistency conditions (21) (22), there exists a unique normalized finitely additive $\mathcal{L}_{\mathcal{H}}$ -valued measure

$$\mathbb{M} : \mathcal{A}_{\Lambda} \rightarrow \mathcal{L}_{\mathcal{H}}, \quad \mathbb{M}(\Lambda) = \mathbb{I}_{\mathcal{H}}, \quad (23)$$

such that

$$\mathbb{M} \left(\pi_{(t_1, \dots, t_n)}^{-1}(F) \right) = \mathfrak{M}_{(t_1, \dots, t_n)}(F) \quad (24)$$

for all sets $F \in \mathcal{F}_{\Lambda_{t_1} \times \dots \times \Lambda_{t_n}}$ and an arbitrary finite index collection $\{t_1, \dots, t_n\} \subset T$.

Note that family (13) of the product measures $\mathcal{P}_{(X_1, \dots, X_n)}$ represents a particular example of a family (20) if, in the latter, we replace

$$\begin{aligned} T &\rightarrow \mathfrak{X}_{\mathcal{H}}, & \Lambda_t &\rightarrow \text{sp}X, \\ \mathcal{F}_t &\rightarrow \mathcal{B}_{\text{sp}X}, & \mathcal{F}_{\Lambda_{t_1} \times \dots \times \Lambda_{t_n}} &\rightarrow \mathcal{F}_{\text{sp}X_1 \times \dots \times \text{sp}X_n}. \end{aligned} \quad (25)$$

Moreover, in view of lemma 1, all measures $\mathcal{P}_{(X_1, \dots, X_n)}$ satisfy the consistency conditions (21), (22).

Therefore, similarly to our notations in lemma 2, we denote by let $\tilde{\Lambda} := \prod_{X \in \mathfrak{X}_{\mathcal{H}}} \text{sp}X$ the set of all real-valued functions $\tilde{\lambda} : \mathfrak{X}_{\mathcal{H}} \rightarrow \cup_{X \in \mathfrak{X}_{\mathcal{H}}} \text{sp}X$ with values $\tilde{\lambda}_X := \tilde{\lambda}(X) \in \text{sp}X$.

Let

$$\begin{aligned} \pi_{(X_1, \dots, X_n)}(\tilde{\lambda}) &: = (\pi_{X_1}(\tilde{\lambda}), \dots, \pi_{X_n}(\tilde{\lambda})) \in \text{sp}X_1 \times \dots \times \text{sp}X_n \subseteq \mathbb{R}^n, \\ \pi_X(\tilde{\lambda}) &: = \tilde{\lambda}_X \in \text{sp}X. \end{aligned} \quad (26)$$

be the canonical projection $\tilde{\Lambda} \rightarrow \text{sp}X_1 \times \dots \times \text{sp}X_n$. The set

$$\mathcal{A}_{\tilde{\Lambda}} := \{ \pi_{(t_1, \dots, t_n)}^{-1}(F) \subseteq \tilde{\Lambda} \mid F \in \mathcal{F}_{\text{sp}X_1 \times \dots \times \text{sp}X_n}, \{X_1, \dots, X_n\} \subset \mathfrak{X}_{\mathcal{H}}, n \in \mathbb{N} \}. \quad (27)$$

of all cylindrical subsets $\pi_{(t_1, \dots, t_n)}^{-1}(F)$ constitutes an algebra on $\tilde{\Lambda}$.

Lemma 2 implies.

Theorem 1 For family (13) of finitely additive measures $\mathcal{P}_{(X_1, \dots, X_n)}$, there exists a unique normalized finitely additive $\mathcal{L}_{\mathcal{H}}^{(s)}$ -valued measure

$$\mathbb{P} : \mathcal{A}_{\tilde{\Lambda}} \rightarrow \mathcal{L}_{\mathcal{H}}^{(s)}, \quad \mathbb{P}(\tilde{\Lambda}) = \mathbb{I}_{\mathcal{H}}, \quad (28)$$

such that

$$\mathbb{P} \left(\pi_{(X_1, \dots, X_n)}^{-1}(F) \right) = \mathcal{P}_{(X_1, \dots, X_n)}(F) \quad (29)$$

for all sets $F \in \mathcal{F}_{\text{sp}X_1 \times \dots \times \text{sp}X_n}$ and an arbitrary finite collection $\{X_1, \dots, X_n\} \subset \mathfrak{X}_{\mathcal{H}}$. In particular,

$$\begin{aligned} \mathbb{P}(\pi_{X_1}^{-1}(B_1) \cap \dots \cap \pi_{X_n}^{-1}(B_n)) &= \frac{1}{n!} \{P_{X_1}(B_1) \cdot \dots \cdot P_{X_n}(B_n)\}_{\text{sym}}, \\ B_1 &\in \mathcal{B}_{\text{sp}X_1}, \dots, B_n \in \mathcal{B}_{\text{sp}X_n} \end{aligned} \quad (30)$$

for each finite number of mutually non-equal operators $X_1, \dots, X_n \in \mathfrak{X}_{\mathcal{H}}$.

Theorem 1 implies.

Proposition 1 *Let $\{\mathcal{P}_{(X_1, \dots, X_n)}\}$ be $\mathcal{L}_{\mathcal{H}}^{(s)}$ -valued measures (10). For every density operator ρ on \mathcal{H} , there exists a unique normalized finitely additive real-valued measure*

$$\mu_\rho : \mathcal{A}_{\tilde{\Lambda}} \rightarrow \mathbb{R}, \quad \mu_\rho(\tilde{\Lambda}) = 1, \quad (31)$$

such that

$$\text{tr}[\rho \mathcal{P}_{(X_1, \dots, X_n)}(F)] = \mu_\rho \left(\pi_{(X_1, \dots, X_n)}^{-1}(F) \right) \quad (32)$$

for all sets $F \in \mathcal{F}_{\text{sp}X_1 \times \dots \times \text{sp}X_n}$ and an arbitrary finite collection $\{X_1, \dots, X_n\} \subset \mathfrak{X}_{\mathcal{H}}$. In particular,

$$\begin{aligned} \frac{1}{n!} \text{tr}[\rho \{P_{X_1}(B_1) \cdot \dots \cdot P_{X_n}(B_n)\}_{\text{sym}}] &= \mu_\rho \left(\pi_{X_1}^{-1}(B_1) \cap \dots \cap \pi_{X_n}^{-1}(B_n) \right), \\ B_1 &\in \mathcal{B}_{\text{sp}X_1}, \dots, B_n \in \mathcal{B}_{\text{sp}X_n}, \end{aligned} \quad (33)$$

for each finite number of mutually non-equal operators $X_1, \dots, X_n \in \mathfrak{X}_{\mathcal{H}}$.

Proof. For a density operator ρ on \mathcal{H} , relation (29) implies

$$\text{tr}[\rho \mathcal{P}_{(X_1, \dots, X_n)}(F)] = \text{tr}[\rho \mathbb{P} \left(\pi_{(X_1, \dots, X_n)}^{-1}(F) \right)] \quad (34)$$

for all sets $F \in \mathcal{F}_{\text{sp}X_1 \times \dots \times \text{sp}X_n}$. Introduce on the algebra $\mathcal{A}_{\tilde{\Lambda}}$ the normalized finitely additive real-valued measure

$$\mu_\rho(A) := \text{tr}[\rho \mathbb{P}(A)], \quad A \in \mathcal{A}_{\tilde{\Lambda}}. \quad (35)$$

Since \mathbb{P} is a unique $\mathcal{L}_{\mathcal{H}}^{(s)}$ -valued finitely additive measure on $\mathcal{A}_{\tilde{\Lambda}}$ satisfying condition (29), the measure μ_ρ defined by relation (34) is also a unique normalized real-valued finitely additive measure on $\mathcal{A}_{\tilde{\Lambda}}$ satisfying condition (32), hence, (33). ■

5 Quasi-classical probability modelling

Based on theorem 1 and proposition 1, we proceed to prove that the Hilbert space description of all joint von Neumann measurements on a quantum system can be reproduced via a set of random variables and a set of normalized real-valued measures on a single space $(\Omega, \mathcal{F}_\Omega)$.

Theorem 2 *Let \mathcal{H} be a complex separable Hilbert space. There exist:*

- (i) *a set Ω and an algebra \mathcal{F}_Ω of subsets of Ω ;*
- (ii) *a $\mathcal{F}_\Omega/\mathcal{B}_{\text{sp}X}$ -measurable real-valued function (random variable) $f_X : \Omega \rightarrow \text{sp}X$ for each quantum observable X on \mathcal{H} ;*

such that $f_{X_1} \neq f_{X_2}$ for $X_1 \neq X_2$ and, to each quantum state ρ on \mathcal{H} , there corresponds a unique normalized finitely additive real-valued measure μ_ρ on $(\Omega, \mathcal{F}_\Omega)$ satisfying the relation

$$\begin{aligned} \frac{1}{n!} \text{tr}[\rho \{P_{X_1}(B_1) \cdot \dots \cdot P_{X_n}(B_n)\}_{\text{sym}}] &= \mu_\rho \left(f_{X_1}^{-1}(B_1) \cap \dots \cap f_{X_n}^{-1}(B_n) \right), \\ B_1 &\in \mathcal{B}_{\text{sp}X_1}, \dots, B_n \in \mathcal{B}_{\text{sp}X_n}, \end{aligned} \quad (36)$$

for each finite collection $\{X_1, \dots, X_n\}$ of quantum observables on \mathcal{H} . In particular,

$$\begin{aligned} \text{tr}[\rho\{P_{X_1}(B_1) \cdot \dots \cdot P_{X_n}(B_n)\}] &= \mu_\rho \left(f_{X_1}^{-1}(B_1) \cap \dots \cap f_{X_n}^{-1}(B_n) \right), \\ B_1 &\in \mathcal{B}_{\text{sp}X_1}, \dots, B_n \in \mathcal{B}_{\text{sp}X_n}, \end{aligned} \quad (37)$$

for every state ρ and an arbitrary finite collection $\{X_1, \dots, X_n\}$ of quantum observables on \mathcal{H} mutually commuting in the sense of relation (6).

Proof. In order to prove the existence point of theorem 2, let us take the space $(\tilde{\Lambda}, \mathcal{A}_{\tilde{\Lambda}})$ considered in theorem 1. Namely, $\tilde{\Lambda}$ is the set of all real-valued functions $\tilde{\lambda} : \mathfrak{X}_{\mathcal{H}} \rightarrow \cup_{X \in \mathfrak{X}_{\mathcal{H}}} \text{sp}X$ with values $\tilde{\lambda}_X \in \text{sp}X$ and $\mathcal{A}_{\tilde{\Lambda}}$ is the algebra (27) of all cylindrical subsets of $\tilde{\Lambda}$ having the form $\pi_{(X_1, \dots, X_n)}^{-1}(F)$, where $F \in \mathcal{F}_{\text{sp}X_1 \times \dots \times \text{sp}X_n}$ and $\{X_1, \dots, X_n\} \subset \mathfrak{X}_{\mathcal{H}}$.

For each observable $X \in \mathfrak{X}_{\mathcal{H}}$, we take on $(\tilde{\Lambda}, \mathcal{A}_{\tilde{\Lambda}})$ the random variable $\pi_X(\tilde{\lambda}) = \tilde{\lambda}_X \in \text{sp}X \subseteq R$ and note that $\pi_{X_1} \neq \pi_{X_2}$ for $X_1 \neq X_2$.

Then, by proposition 2, to each quantum state ρ on \mathcal{H} , there corresponds a unique normalized real-valued measures μ_ρ on $\mathcal{A}_{\tilde{\Lambda}}$ satisfying (33) and, hence, relations (36) and (37).

■

From relations (36) and (2) it follows.

Corollary 1 *In theorem 2, let $\{X_1, \dots, X_n\}$ be a finite collection of bounded quantum observables X_1, \dots, X_n on \mathcal{H} . Then*

$$\frac{1}{n!} \text{tr}[\rho\{X_1 \cdot \dots \cdot X_n\}_{\text{sym}}] = \int_{\Omega} f_{X_1}(\omega) \cdot \dots \cdot f_{X_n}(\omega) \mu_\rho(d\omega) \quad (38)$$

for all quantum states ρ .

Theorem 2 implies.

Corollary 2 *For the probabilistic description of all joint von Neumann measurements upon a quantum state ρ on a complex separable Hilbert space \mathcal{H} , there exist:*

(i) *a measure space $(\Omega, \mathcal{F}_{\Omega}, \mu_\rho)$, where \mathcal{F}_{Ω} is an algebra of subsets of a set Ω and μ_ρ is a normalized finitely additive real-valued measure on \mathcal{F}_{Ω} ;*

(ii) *a random variable $f_X : \Omega \rightarrow \text{sp}X$ for each quantum observable X on \mathcal{H} ;*

such that $f_{X_1} \neq f_{X_2}$ for $X_1 \neq X_2$, a space $(\Omega, \mathcal{F}_{\Omega})$ and random variables $\{f_X\}$ do not depend on a state ρ and the representation

$$\begin{aligned} \text{tr}[\rho\{P_{X_1}(B_1) \cdot \dots \cdot P_{X_n}(B_n)\}] &= \mu_\rho \left(f_{X_1}^{-1}(B_1) \cap \dots \cap f_{X_n}^{-1}(B_n) \right), \\ B_1 &\in \mathcal{B}_{\text{sp}X_1}, \dots, B_n \in \mathcal{B}_{\text{sp}X_n}, \end{aligned} \quad (39)$$

holds for an arbitrary finite collection $\{X_1, \dots, X_n\}$ of quantum observables on \mathcal{H} mutually commuting in the sense of relation (6).

From corollary 2 it follows that the probability distributions of *all joint* von Neumann measurements on a quantum state ρ can be reproduced in terms of a single measure space $(\Omega, \mathcal{F}_{\Omega}, \mu_\rho)$ with a normalized real-valued measure μ_ρ and a set of random variables, each modelling the corresponding quantum observable in all joint von Neumann measurements and depending only on this quantum observable.

Representation (39) can be otherwise expressed in the form

$$\mathrm{tr}[\rho\{P_{X_1}(B_1) \cdot \dots \cdot P_{X_n}(B_n)\}] = \int_{\Omega} \chi_{f_{X_1}^{-1}(B_1)}(\omega) \cdot \dots \cdot \chi_{f_{X_n}^{-1}(B_n)}(\omega) \mu_{\rho}(\mathrm{d}\omega), \quad (40)$$

which is specific for joint probability distributions in a local quasi hidden variable (LqHV) model of the deterministic type, see Refs. [7, 8].

Thus, all joint von Neumann measurements on a finite dimensional quantum state admit a *deterministic quasi hidden variable (qHV) model* [7, 8] and, in this model, a random variable modelling a quantum observable depends only on this quantum observable.

The following statement is proved in appendix C.

Proposition 2 *In theorem 2:*

- (i) *If $f_X : \Omega \rightarrow \mathrm{sp}X$ is a random variable modelling a quantum observable X via representation (36), then, for each Borel function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, the random variable $\varphi \circ f_X : \Omega \rightarrow \mathrm{sp}\varphi(X)$ models the quantum observable $\varphi(X)$;*
- (ii) *If μ_{ρ_k} , $k = 1, \dots, K < \infty$, are normalized real-valued measures, each uniquely corresponding to a quantum state ρ_k via representation (36), then the measure $\sum \alpha_k \mu_{\rho_k}$, with $\alpha_k > 0$, $\sum \alpha_k = 1$, uniquely corresponds to the state $\sum \alpha_k \rho_k$.*

From theorem 2 and proposition 2 it follows that the observable $\varphi(X)$ is modeled via representation (36) by either of two random variables $\varphi \circ f_X$ or $f_{\varphi(X)}$ on $(\Omega, \mathcal{F}_{\Omega})$ and, for arbitrary X and φ , the latter random variables do not need to coincide.

Consider, for example, the random variables π_X , $X \in \mathfrak{X}_{\mathcal{H}}$, defined on the space $(\tilde{\Lambda}, \mathcal{A}_{\tilde{\Lambda}})$ by relation (26) and used by us above for the proof of the existence point of theorem 2. The random variables $\varphi \circ \pi_X$ and $\pi_{\varphi(X)}$ do not need to coincide for all observable X and all Borel functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$. Hence, for arbitrary X and φ , the observable $\varphi(X)$ is equivalently modeled on the space $(\tilde{\Lambda}, \mathcal{A}_{\tilde{\Lambda}})$ by either of two different random variables – $\varphi \circ \pi_X$ and $\pi_{\varphi(X)}$.

Note also that, according to the Kochen and Specker theorem [13], for a Hilbert space \mathcal{H} of a dimension $d \geq 3$, there does not exist a space $(\Omega, \mathcal{F}_{\Omega})$, where, under the condition $\varphi \circ f \stackrel{\Phi}{\mapsto} \varphi \circ X$, $\forall \varphi$, a mapping $f \stackrel{\Phi}{\mapsto} X$ from a set of random variables on $(\Omega, \mathcal{F}_{\Omega})$ onto the set of all quantum observables on \mathcal{H} could be one-to-one.

6 Conclusions

In the present paper, we have introduced (lemma 2, theorem 1) a generalization of some items of the Kolmogorov extension theorem [2, 12] to the case of consistent operator-valued measures and, based on this, we have proved (theorem 2) that the Hilbert space description of all joint von Neumann measurements on a quantum system can be reproduced in terms of a single space $(\Omega, \mathcal{F}_{\Omega})$ via a set of normalized real-valued measures, each uniquely corresponding to some quantum state, and a set of random variables, each being determined only by a modelled quantum observable and such that if f_X is a random variable modelling a quantum observable X via representation (36), then, for each Borel function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, the random variable $\varphi \circ f_X$ models (proposition 2) the observable $\varphi \circ X$.

This result, in particular, means that *all joint von Neumann measurements* on a quantum state ρ admit (corollary 2) a *deterministic quasi hidden variable (qHV) model* [7, 8] and, in

this model, a random variable modelling a quantum observable depends only on this quantum observable.

From the probabilistic point of view, a deterministic qHV model constitutes *the quasi-classical probability model* – a new general probability model formulated in Ref. [8].

In the quasi-classical probability model specified by a measure space $(\Omega, \mathcal{F}_\Omega, \nu)$, a normalized real-valued measure ν does not need to be positive but:

- (i) an observable with a value space $(\Lambda, \mathcal{F}_\Lambda)$ is represented only by such a random variable $f : \Omega \rightarrow \Lambda$, for which the normalized measure $\nu(f^{-1}(\cdot))$ on \mathcal{F}_Λ is a probability one;
- (ii) a joint measurement of two observables represented by random variables f_1, f_2 with value spaces $(\Lambda_n, \mathcal{F}_{\Lambda_n})$, $n = 1, 2$, is possible iff $\nu(f_1^{-1}(F_1) \cap f_2^{-1}(F_2)) \geq 0$, for all $F_1 \in \mathcal{F}_{\Lambda_1}$, $F_2 \in \mathcal{F}_{\Lambda_2}$.

We stress that though, in the quasi-classical probability model, a measure space $(\Omega, \mathcal{F}_\Omega, \nu)$ does not need to be a probability one, each modelled measurement, single or joint, satisfies the Kolmogorov axioms [2] in the sense that it is described by a probability space, where a probability measure is specified in the above item (i) or (ii).

From the above results it also follows that if joint von Neumann measurements on an N -partite quantum state ρ are performed by space-like separated parties, then, for these joint measurements, there exists a quasi-classical probability model $(\Omega, \mathcal{F}_\Omega, \mu_\rho)$, which is *local* in the sense that each party marginal measurement is described by a random variable depending only on the corresponding observable at the corresponding site, but does not need to be "classical" – in the sense of positivity a measure μ_ρ .

From the conceptual point of view, the latter mathematical result not only supports our arguments [6, 14] on a difference between Bell's locality and the EPR locality but also directly points to a misleading character of Bell's conjecture on quantum "non-locality" (*action on a distance*) that was introduced by Bell [15, 16] only in view of non-existence of a local classical probability model for spin measurements on the two-qubit singlet and due to his further choice that this non-existence is caused precisely by violation of "locality" but not by violation of "classicality".

7 Appendix A

Relation (14) follows explicitly from the symmetrized form of the right-hand side of condition (11).

For clearness, let us first prove relation (15) for a collection $\{X_1, \dots, X_n\}$ of bounded quantum observables with discrete spectrums. In this case, the measure $\mathcal{P}_{(X_1, \dots, X_n)}$ is given by representation (12) and taking into the account the relation $P_{X_i}(\text{sp}X_i) = \mathbb{I}_{\mathcal{H}}$, we have:

$$\begin{aligned}
& \mathcal{P}_{(X_1, \dots, X_n)}(\{(x_1, \dots, x_n) \in \text{sp}X_1 \times \dots \times \text{sp}X_n \mid (x_{i_1}, \dots, x_{i_k}) \in F\}) \quad (\text{A1}) \\
&= \frac{1}{n!} \sum_{(\lambda_{X_{i_1}}, \dots, \lambda_{X_{i_k}}) \in F} \{P_{X_1}(\{\lambda_{X_1}\}) \cdot \dots \cdot P_{X_n}(\{\lambda_{X_n}\})\}_{\text{sym}} \\
&= \frac{1}{k!} \sum_{(\lambda_{X_{i_1}}, \dots, \lambda_{X_{i_k}}) \in F} \{P_{X_{i_1}}(\{\lambda_{X_{i_1}}\}) \cdot \dots \cdot P_{X_{i_k}}(\{\lambda_{X_{i_k}}\})\}_{\text{sym}} \\
&= \mathcal{P}_{(X_{i_1}, \dots, X_{i_k})}(F).
\end{aligned}$$

In order to prove (15) for an arbitrary collection $\{X_1, \dots, X_n\} \subset \mathfrak{X}_{\mathcal{H}}$ of quantum observables, let us denote by \mathcal{E} the set of all rectangles $E := B_{i_1} \times \dots \times B_{i_k}$ with sides $B_i \in \mathcal{B}_{\text{sp}X_i}$.

Since the algebra $\mathcal{F}_{\text{sp}X_{i_1} \times \dots \times \text{sp}X_{i_k}}$ consists of all finite unions of mutually disjoint rectangles from \mathcal{E} , every $F \in \mathcal{F}_{\text{sp}X_{i_1} \times \dots \times \text{sp}X_{i_k}}$ admits a finite decomposition

$$F = \bigcup_{m=1, \dots, M} E_m, \quad E_{m_1} \cap E_{m_2} = \emptyset, \quad E_m \in \mathcal{E}, \quad M < \infty. \quad (\text{A2})$$

Taking into the account that $\mathcal{P}_{(X_1, \dots, X_n)}$, $\mathcal{P}_{(X_{i_1}, \dots, X_{i_k})}$ are measures and relations (A2), (11) and $\text{P}_{X_i}(\text{sp}X_i) = \mathbb{I}_{\mathcal{H}}$, we derive:

$$\begin{aligned} & \mathcal{P}_{(X_1, \dots, X_n)}(\{(x_1, \dots, x_n) \in \text{sp}X_1 \times \dots \times \text{sp}X_n \mid (x_{i_1}, \dots, x_{i_k}) \in F\}) \\ &= \sum_m \mathcal{P}_{(X_1, \dots, X_n)}(\{(x_1, \dots, x_n) \in \text{sp}X_1 \times \dots \times \text{sp}X_n \mid (x_{i_1}, \dots, x_{i_k}) \in E_m\}) \\ &= \sum_m \frac{1}{k!} \left\{ \text{P}_{X_{i_1}}(B_{i_1}^{(m)}) \cdot \dots \cdot \text{P}_{X_{i_k}}(B_{i_k}^{(m)}) \right\}_{\text{sym}} = \sum_m \mathcal{P}_{(X_{i_1}, \dots, X_{i_k})}(E_m) \\ &= \mathcal{P}_{(X_{i_1}, \dots, X_{i_k})}(F). \end{aligned} \quad (\text{A3})$$

This prove lemma 1.

8 Appendix B

Our proof of lemma 2 is quite similar to the proof [2, 12] of the corresponding items in the Kolmogorov extension theorem for consistent probability measures (16).

Let \mathcal{A}_Λ be the algebra on Λ defined by relations (17) - (19). For a set $A \in \mathcal{A}_\Lambda$ admitting the representation $A = \pi_{(t_1, \dots, t_n)}^{-1}(F)$, where $\{t_1, \dots, t_n\} \subset T$ and $F \in \mathcal{F}_{\Lambda_{t_1} \times \dots \times \Lambda_{t_n}}$, we let

$$\mathbb{M}(A) := \mathfrak{M}_{(t_1, \dots, t_n)}(F). \quad (\text{B1})$$

In order to show that relation (B1) defines correctly a set function on \mathcal{A}_Λ , we must prove that this relation implies a unique value to a set $A \in \mathcal{A}_\Lambda$ even if this set admit two different representations, say:

$$\begin{aligned} A &= \pi_{(t_{i_1}, \dots, t_{i_k})}^{-1}(F) \equiv \left\{ \lambda \in \Lambda \mid (\lambda_{t_{i_1}}, \dots, \lambda_{t_{i_k}}) \in F \right\}, \\ A &= \pi_{(t_{j_1}, \dots, t_{j_m})}^{-1}(F') \equiv \left\{ \lambda \in \Lambda \mid (\lambda_{t_{j_1}}, \dots, \lambda_{t_{j_m}}) \in F' \right\} \end{aligned} \quad (\text{B2})$$

for some sets $F \in \mathcal{F}_{\Lambda_{t_{i_1}} \times \dots \times \Lambda_{t_{i_k}}}$ and $F' \in \mathcal{F}_{\Lambda_{t_{j_1}} \times \dots \times \Lambda_{t_{j_m}}}$ and some index collections $\{t_{i_1}, \dots, t_{i_k}\}$, $\{t_{j_1}, \dots, t_{j_m}\} \subset T$.

Denote

$$\{t_{i_1}, \dots, t_{i_k}\} \cup \{t_{j_1}, \dots, t_{j_m}\} := \{t_1, \dots, t_n\}. \quad (\text{B3})$$

From (42) it follows that sets F and F' are such that, for a point in $\Lambda_{t_1} \times \dots \times \Lambda_{t_n}$, the condition $(\lambda_{i_1}, \dots, \lambda_{i_k}) \in F$ implies the condition $(\lambda_{j_1}, \dots, \lambda_{j_m}) \in F'$ and vice versa, that is:

$$\begin{aligned} & \{ (\lambda_1, \dots, \lambda_n) \in \Lambda_{t_1} \times \dots \times \Lambda_{t_n} \mid (\lambda_{i_1}, \dots, \lambda_{i_k}) \in F \} \\ &= \{ (\lambda_1, \dots, \lambda_n) \in \Lambda_{t_1} \times \dots \times \Lambda_{t_n} \mid (\lambda_{j_1}, \dots, \lambda_{j_m}) \in F' \}. \end{aligned} \quad (\text{B4})$$

In view of relations (B1) - (42) and the consistency conditions (21), (22), we have:

$$\begin{aligned}
\mathbb{M}(\pi_{(t_{i_1}, \dots, t_{i_k})}^{-1}(F)) &= \mathfrak{M}_{(t_{i_1}, \dots, t_{i_k})}(F) \\
&= \mathfrak{M}_{(t_1, \dots, t_n)}(\{(\lambda_1, \dots, \lambda_n) \in \Lambda_{t_1} \times \dots \times \Lambda_{t_n} \mid (\lambda_{i_1}, \dots, \lambda_{i_k}) \in F\}) \\
&= \mathfrak{M}_{(t_1, \dots, t_n)}(\{(\lambda_1, \dots, \lambda_n) \in \Lambda_{t_1} \times \dots \times \Lambda_{t_n} \mid (\lambda_{j_1}, \dots, \lambda_{j_m}) \in F'\}) \\
&= \mathfrak{M}_{(t_{j_1}, \dots, t_{j_m})}(F') \\
&= \mathbb{M}(\pi_{(t_{j_1}, \dots, t_{j_m})}^{-1}(F')).
\end{aligned} \tag{B5}$$

Thus, relation (B1) defines a unique set function $\mathbb{M} : \mathcal{A}_\Lambda \rightarrow \mathcal{L}_\mathcal{H}$ satisfying condition (23).

Since $\Lambda = \pi_{(t_1, \dots, t_n)}^{-1}(\Lambda_{t_1} \times \dots \times \Lambda_{t_n})$ and $\mathfrak{M}_{(t_1, \dots, t_n)}(\Lambda_{t_1} \times \dots \times \Lambda_{t_n}) = \mathbb{I}_\mathcal{H}$, from (B1) it follows that the set function \mathbb{M} is normalized, that is, $\mathbb{M}(\Lambda) = \mathbb{I}_\mathcal{H}$.

In order to prove that the normalized set function $\mathbb{M} : \mathcal{A}_\Lambda \rightarrow \mathcal{L}_\mathcal{H}$ is additive, let us consider in the algebra \mathcal{A}_Λ two disjoint sets

$$A_1 = \pi_{(t_{i_1}, \dots, t_{i_k})}^{-1}(F_1), \quad A_2 = \pi_{(t_{j_1}, \dots, t_{j_m})}^{-1}(F_2), \tag{B6}$$

specified by some index collections $\{t_{i_1}, \dots, t_{i_k}\}, \{t_{j_1}, \dots, t_{j_m}\} \subseteq \{t_1, \dots, t_n\} \subset T$ and sets $F_1 \in \mathcal{F}_{\Lambda_{t_{i_1}} \times \dots \times \Lambda_{t_{i_k}}}$ and $F_2 \in \mathcal{F}_{\Lambda_{t_{j_1}} \times \dots \times \Lambda_{t_{j_m}}}$.

Since $A_1 \cap A_2 = \emptyset$, the sets F_1, F_2 in (B6) are such that, for a point in $\Lambda_{t_1} \times \dots \times \Lambda_{t_n}$, conditions $(\lambda_{i_1}, \dots, \lambda_{i_k}) \in F_1$ and $(\lambda_{j_1}, \dots, \lambda_{j_m}) \in F_2$ are mutually exclusive, that is:

$$\begin{aligned}
&\{(\lambda_1, \dots, \lambda_n) \in \Lambda_{t_1} \times \dots \times \Lambda_{t_n} \mid (\lambda_{i_1}, \dots, \lambda_{i_k}) \in F_1\} \\
&\cap \{(\lambda_1, \dots, \lambda_n) \in \Lambda_{t_1} \times \dots \times \Lambda_{t_n} \mid (\lambda_{j_1}, \dots, \lambda_{j_m}) \in F_2\} \\
&= \emptyset.
\end{aligned} \tag{B7}$$

Taking into the account relations (B1), (B6), (42), the consistency conditions (21), (22) and also that each $\mathfrak{M}_{(t_1, \dots, t_n)}$ is a finitely additive measure, we derive

$$\begin{aligned}
&\mathbb{M}(A_1 \cup A_2) \\
&= \mathfrak{M}_{(t_1, \dots, t_n)}(\{(\lambda_1, \dots, \lambda_n) \in \Lambda_{t_1} \times \dots \times \Lambda_{t_n} \mid (\lambda_{i_1}, \dots, \lambda_{i_k}) \in F_1 \text{ or } (\lambda_{j_1}, \dots, \lambda_{j_m}) \in F_2\}) \\
&= \mathfrak{M}_{(t_1, \dots, t_n)}(\{(\lambda_1, \dots, \lambda_n) \in \Lambda_{t_1} \times \dots \times \Lambda_{t_n} \mid (\lambda_{i_1}, \dots, \lambda_{i_k}) \in F_1\}) \\
&\quad + \mathfrak{M}_{(t_1, \dots, t_n)}(\{(\lambda_1, \dots, \lambda_n) \in \Lambda_{t_1} \times \dots \times \Lambda_{t_n} \mid (\lambda_{j_1}, \dots, \lambda_{j_m}) \in F_2\}) \\
&= \mathfrak{M}_{(t_{i_1}, \dots, t_{i_k})}(F_1) + \mathfrak{M}_{(t_{j_1}, \dots, t_{j_m})}(F_2) \\
&= \mathbb{M}(A_1) + \mathbb{M}(A_2).
\end{aligned} \tag{B8}$$

Hence, the normalized set function \mathbb{M} on \mathcal{A}_Λ defined by relation (B1) is additive and is, therefore, a finitely additive measure on \mathcal{A}_Λ , see remark 1.

Thus, we have proved that the set function $\mathbb{M} : \mathcal{A}_\Lambda \rightarrow \mathcal{L}_\mathcal{H}$ defined by relation (B1) constitutes a unique normalized finitely additive $\mathcal{L}_\mathcal{H}$ -valued measure on the algebra \mathcal{A}_Λ satisfying relation (23).

9 Appendix C

For a complex Hilbert space \mathcal{H} , let $(\Omega, \mathcal{F}_\Omega)$ be specified in theorem 2. Then the representation

$$\begin{aligned}
\frac{1}{n!} \text{tr}[\rho\{P_{X_1}(B_1) \cdot \dots \cdot P_{X_n}(B_n)\}_{\text{sym}}] &= \mu_\rho \left(f_{X_1}^{-1}(B_1) \cap \dots \cap f_{X_n}^{-1}(B_n) \right), \\
B_1 &\in \mathcal{B}_{\text{sp}X_1}, \dots, B_n \in \mathcal{B}_{\text{sp}X_n},
\end{aligned} \tag{C1}$$

holds for all states ρ and an arbitrary finite number of mutually non-equal quantum observables X_1, \dots, X_n on \mathcal{H} .

From (42) and the relations

$$\begin{aligned}\varphi \circ X &\equiv \varphi(X) := \int \varphi(\lambda) P_X(d\lambda), \\ P_{\varphi(X)}(B) &= P_X(\varphi^{-1}(B)), \quad B \in \mathcal{B}_{\text{sp}\varphi(X)}, \\ \text{sp}\varphi(X) &= \varphi(\text{sp}X),\end{aligned}\tag{C2}$$

it follows that, for a Borel function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and an observable X_1 , the relation

$$\begin{aligned}& \frac{1}{n!} \text{tr}[\rho \{P_{\varphi(X_1)}(B_1) \cdot P_{X_2}(B_2) \cdot \dots \cdot P_{X_n}(B_n)\}_{\text{sym}}] \\&= \frac{1}{n!} \text{tr}[\rho \{P_{X_1}(\varphi^{-1}(B_1)) \cdot \dots \cdot P_{X_n}(B_n)\}_{\text{sym}}] \\&= \mu_\rho(f_X^{-1}(\varphi^{-1}(B_1)) \cap \dots \cap f_{X_n}^{-1}(B_n)) \\&= \mu_\rho((\varphi \circ f_X^{-1})(B_1) \cap \dots \cap f_{X_n}^{-1}(B_n))\end{aligned}\tag{C3}$$

is valid for each Borel function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, all sets $B_1 \in \mathcal{B}_{\text{sp}\varphi(X)}, \dots, B_n \in \mathcal{B}_{\text{sp}X_n}$, all states ρ and an arbitrary finite number of mutually non-equal quantum observables X_2, \dots, X_n on \mathcal{H} . By theorem 2, this proves property (i).

Further, let $\rho_k \mapsto \mu_{\rho_k}$, $k = 1, \dots, K < \infty$. From (36) it follows

$$\begin{aligned}& \frac{1}{n!} \text{tr}[(\sum_k \alpha_k \rho_k) \{P_{X_1}(B_1) \cdot \dots \cdot P_{X_n}(B_n)\}_{\text{sym}}] \\&= (\sum_k \alpha_k \mu_{\rho_k}) (f_{X_1}^{-1}(B_1) \cap \dots \cap f_{X_n}^{-1}(B_n))\end{aligned}\tag{C4}$$

for all $B_1 \in \mathcal{B}_{\text{sp}X_1}, \dots, B_n \in \mathcal{B}_{\text{sp}X_n}$ and each finite collection $\{X_1, \dots, X_n\}$ of quantum observables on \mathcal{H} . By theorem 2, this proves property (ii).

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